On Existence of Solution of Functional Equation $\varphi(x)+\varphi[f(x)]=F(x)$

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Existencia de la solución a la ecuación funcional $\varphi(x)+\varphi[f(x)]=F(x)$

Resumen

En este artículo se considera la Ecuación $\varphi(x)+\varphi[f(x)]=F(x)$ y se dan las preposiciones necesarias para la existencia de más de una solución de la fórmula explícita.

Palabra clave: Formula explícita, ecuación funcional, teórica, prueba.

Abstract

In this paper equation $\varphi(x)+\varphi[f(x)]=F(x)$ is considered and prepositions needed for existence of the most one solution given by explicit formula are given.

Key Words: Explicit formula, functional equation, Theorem, Proof.

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The results of this paper are connected to solutions of the functional equation

$$\varphi(x) + \varphi[f(x)] = F(x)\ , \qquad (1)$$

where $\varphi(x)$ denote unknown function, and f(x) and F(x) are known functions.

Equation (1) is direct generalization of

$$\varphi(x) + \varphi(x^2) = x$$

having been discussed by Steinhaus (1955), or

$$\varphi(x) + \varphi(x^\alpha) = x\ , \qquad (\alpha > 1)$$

solved by Hardy (1949).

Kuczma(1959) proves that under some natural assumptions equation (1) has infinitely many solutions continuous for every x that is not a root of equation

$$f(x) = x\ . \qquad (2)$$

In the same paper, under assumption that solution is continuous for $x = x_0$ satisfying (2), existence of a most such solution is proved.

Some solutions of the equation (1) are examined Malenica (1980a, 1980b) and Pjanic(2006).

In this paper are given additional assumptions needed for existence of the most one solution of equation (1) that can be given by an explicit formula.

Next two theorems are proved in Kuczma(1959).

THEOREM 1. *Let F(x) be continuous and f(x) continuous and strictly increasing in [a,b]. Then (1) has indefinitely many solutions that are continuous in (a,b).*

THEOREM 2. *Let assumptions of Theorem 1 hold. Then (1) has at most one solution continuous in [a,b) and at most one solution continuous in (a,b].*

In the theorems above and troughout this paper it is assumed that

1) f(x) is continuous strongly monotone function on [a,b] and that
   $f(a) = a\ ,\ f(b) = b,\ \ f(x) > x,\ x \in (a,b);$
2) $f^0(x) = x,\ f^{k+1}(x) = f(f^k(x)),\ k \in \{0,\pm1,\pm2,...\}$

THEOREM 3. *If functions f(x) and F(x) satisfy assumptions of Theorem 1 and if there exist functions φ(x) and ψ(x) that fulfill (1) and are continuous in (a,b] and [a,b) respectively, then*

$$\varphi(x) = \frac{1}{2}F(b) + \sum_{k=0}^{\infty}(-1)^k \left[F\left(f^k(x)\right) - F(b)\right] \qquad (3)$$

$$\psi(x) = \frac{1}{2}F(a) - \sum_{k=1}^{\infty}(-1)^k \left[F\left(\varphi^{-k}(x)\right) - F(a)\right] \qquad (4)$$

Proof. At first notice that F(b) = 0. Let $\varphi(x)$ be solution of (1) continuous in (a,b].

Putting x = b in (1) one will obtain $\varphi(b) = 0$.
Whereas $\varphi(x)$ is continuous for x = b there has to exist lim $\varphi(x)=0$ and consequently

$$\lim_{n\to\infty}\varphi\left[f^n(x)\right] = 0\cdot \qquad (5)$$

It is obvious from (1) that

$$\varphi(x) = F(x) - \varphi[f(x)] \qquad (6)$$

$$\varphi[f(x)] = F[f(x)] - \varphi[f^2(x)] \qquad (7)$$

Relations (6) and (7) give

$$\varphi(x) = F(x) - F[f(x)] + \varphi[f^2(x)].$$

Following equation can be obtained by induction

$$\varphi(x)=\sum_{k=0}^{n}(-1)^k F\left[f^k(x)\right]+(-1)^{n+1}\varphi\left[f^{n+1}(x)\right]$$

i.e.

$$\varphi(x)-(-1)^{n+1}\varphi\left[f^{n+1}(x)\right]=\sum_{k=0}^{n}(-1)^k F\left[f^k(x)\right]$$

Changing to limit when $n\to\infty$, according to (5), one gets

$$\varphi(x)=\sum_{k=0}^{\infty}(-1)^k F\left[f^k(x)\right]$$

Now, take arbitrary F(b). Whereas $\varphi(x)$ is solution of (1) continuous in (a,b], function

$$\gamma(x)\overset{def}{=}\varphi(x)-\frac{1}{2}F(b) \qquad (8)$$

is a solution of

$\gamma(x) + \gamma[f(x)] = F(x) - F(b)$

continuous in (a,b].

Hence, $\gamma(x)$ has to be given by

$$\gamma(x)=\sum_{k=0}^{\infty}(-1)^k\left[F\left(f^k(x)\right)-F(b)\right]$$

Therefore, accordingly to (8), expression (3) is obtained.

Similarly (4) can be obtained.

Naturally, there is a possibility that solution of (1) continuous for x = a or x = b does not exist. Existence of such solution depends of F(x).

It will be proved that with some simple presumptions on function F(x) such solution necessarily exists.

THEOREM 4. *If functions f(x) and F(x) satisfy assumptions of Theorem 1 and if F(x) is also monotone in (b-η,b] or [a,a+η), where η and a are positive numbers, then solution of (1) continuous in (a,b] or [a,b) necessarily exists.*

Proof. Suppose F(x) is increasing in (b-η,b]. It will be proved that

$$\sum_{k=0}^{\infty}(-1)^k\left[F\left[f^k(x)\right]-F(b)\right] \qquad (9)$$

converges uniformly in (h,b) for every a < h < b.

Put

$h_n = f^n(h)$.

For as much as $\lim_{n\to\infty}h_n=b$ then there exists whole number N_1 such that $h_n\in(b-\eta,b]$ for $n>N_1$.

Furthermore, for any given $\varepsilon>0$ there is $N>N_1$ such that

$F(b)-F(h_n)<\varepsilon$ for $n>N$.

Take arbitrary $x\in(h,b)$ and put $x_n=f^n(x)$.

For every n there is $x_n\geq h_n$, hence

$F(x_n)\geq F(h_n)$ for $n>N$ and $x\in(b-\eta,b]$.

Obviously

$$\sum_{k=N+1}^{\infty}(-1)^k\left[F\left(f^k(x)\right)-F(b)\right]$$

converges.

More ever, following inequality holds

$$\left|\sum_{k=n}^{\infty}(-1)^k\left[F(x_k)-F(b)\right]\right|\leq F(b)-F(x_k)\leq F(b)-F(h_k)<\varepsilon,\ \ n>N.$$

This implies uniform convergence of (9).

Hence, $\varphi(x)$, defined by (3), is continuous in (a,b].

Obviously $\varphi(x)$ satisfies (1).

Proof of remaining cases can be obtained on similar way.

THEOREM 5. *If functions f(x) and F(x) satisfy assumptions of Theorem1 and if*

$|F(x)-F(b)|\leq G(x)$,

or

$|F(x)-F(a)|\leq G(x)$,

holds in [a,b],

where G(x) is arbitrary bounded function such that

$$\frac{G\left[f(x)\right]}{G(x)}<\nu<1,\ \text{for } x\in(b-\eta,b) \qquad (10)$$

or

$$\frac{G(x)}{G[f(x)]} < v < 1 \text{ , for } x \in (a,a+\eta), \qquad (11)$$

then there exists at most one solution of (1) continuous in (a,b] or [a,b).

Proof. Suppose (10) holds and prove that (9) converges uniformly in [h,b] for every a < h < b.
Put $h_n = f^n(h)$. Then there is whole number N such that $h_n \in (b-\eta, b]$ for n > N.
Put

$$A_n = \begin{cases} \sup\limits_{[h,b]} G(x) & , n \le N \\ \sup\limits_{[h_n,h_{n+1}]} G(x) & , n > N \end{cases}$$

Sequence $\{A_n\}$ is decreasing and, more ever $\sum_{n=0}^{\infty} A_n$ converges.
In fact, $f^{-1}(x) \in [h_n,h_{n+1})$ for every $x \in [h_{n+1},h_{n+2}]$.
Thus, according to (10),

$$G(x) < v\, G[f^{-1}(x)] < v \sup_{[h_n,h_{n+1}]} G(x) = vA_n \quad \text{for } n > N.$$

Thereafter,

$$A_{n+1} = \sup_{[h_{n+1},h_{n+2}]} G(x) \le vA_n \quad \text{for } n > N,$$

That implies convergence of $\sum_{n=0}^{\infty} A_n$.

Take now arbitrary $x \in [h,b]$ and put $x_n = f^n(x)$.

Then

$$|F(x_n)-F(b)| \le G(x_n).$$

As $x_n \ge h_n$, then there is whole number $k \ge 0$ such that $x_n \in [h_{n+k}, h_{n+k+1})$.
So, we get $G(x_n) \le A_{n+k} \le A_n$.
Hence,

$$|F(x_n)-F(b)| \le A_n , \; x \in [h, b]$$

that implies uniform convergence of (14).
Therefore, $\varphi(x)$ defined with (3) is continuous in (a, b]. Obviously $\varphi(x)$ fulfills (1).

Relation (11) can be proved similarly.

REMARK: It is important to note that all results are valid if one or both ends of modulus interval (a,b) are infinite. If, b = ∞ then we consider $F(b) = \lim_{b\to\infty} F(b)$; and $\varphi(x)$ is said to be continuous in infinity if there exist $\lim_{x\to\infty}\varphi(x) < \infty$. Similar is for a = ∞.

However, if $\lim_{x\to b} F(x) = \infty$ (b = ∞ or b < ∞), then solutions of (1) such that $\lim_{x\to b}\varphi(x)$ exists and $\lim_{x\to b}\varphi(x) = \infty$, are not the only one solutions of (1).

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