On conditions for existence of the one and only one solution of functional equation φ(x)+φ[f(x)]=F(x)

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Resumen

In this paper equation **φ**(x) +φ [f(x)] =F(x) is considered and prepositions needed for existence of one and only one solution of this equation on the given interval.

Key words: functional equation, explicit formula, proof, sumability.

Condiciones para la existencia de una y sólo una solución de la ecuación funcional φ(x)+φ[f(x)]=F(x)

Resumen

En este artículo se considera la ecuación φ(x) +φ [f(x)] =F(x) y se dan las preposiciones necesarias para la existencia de una única solución de la ecuación en el intervalo dado.

Palabras clave: ecuación funcional, fórmula explicita, prueba, sumabilidad.

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The results of this paper are related to conditions under what functional equation

$$\varphi(x) + \varphi[f(x)] = F(x) , \qquad (1)$$

where $\varphi(x)$ denote unknown function, and f(x) and F(x) are known functions, has one and only one solution.

Kuczma (1959) proves that under some natural assumptions equation (1) has infinitely many solutions continuous for every x that is not a root of equation

$$f(x) = x . \qquad (2)$$

In the same paper, under assumption that solution is continuous for $x = x_0$ satisfying (2), existence of a most such solution is proved.

Some solutions of the equation (1) were examined by Malenica (1980a, 1980b) and Pjanic (2006, 2007). Pjanic (2007) gives assumptions needed for existence of the most one solution of equation (1) that can be given by an explicit formula.

Next theorem is proved in Kuczma (1959)
THEOREM 1. Given (1). If *F(x) is continuous and if f(x) is continuous and strictly increasing in [a,b] and if there exist functions $\varphi(x)$ and $\psi(x)$ that fulfill (1) and are continuous in (a,b] and [a,b) respectively, then*

$$\varphi(x)=\frac{1}{2}F(b)+\sum_{k=0}^{\infty}(-1)^k\Big[F\big(f^k(x)\big)-F(b)\Big] \qquad (3)$$

$$\psi(x)=\frac{1}{2}F(a)-\sum_{k=1}^{\infty}(-1)^k\Big[F\big(\varphi^{-k}(x)\big)-F(a)\Big] \qquad (4)$$

In this paper will be presented results on some of the features of the series (3). Those results are connected to sumability of the series (3) as well as the proof of the existence of the only one solution of the equation (1) that is continuous in the interval (a, b].

Those results are given in the Theorems 2, 4, 5 and 6.

Theorem 2 and Theorem 4, consider different cases of sumability of series (3), and give sufficient conditions for existence of the only one solution of (1). Theorem 5 and Theorem 6 give sufficient conditions for existence of the only one continuous solution of (1) in (a, b].

Results on the series (4) can be obtained analogicaly.

The next assumptions will be valid in the all following theorems.

1. Function f(x) is continuous, strictly increasing in the interval [a, b], and
   f(a) = a,
   f(b) = b,
   $f(x) > x,\ x \in (a, b)$.
2. $f^0(x) = x$,
   $f^{\nu+1}(x) = f[f^{\nu}(x)], \quad \nu = 0, \pm 1, \pm 2, \ldots$

THEOREM 2. Let *function F(x) be defined in [a, b].*
a) If series (3) converges in [a, b],then its sum $\varphi(x)$ is solution of equation (1).

b) If series (3) is T-sumabile with the sum $\varphi(x)$, where T is a matrix

$$T=(a_{kn}) = \begin{bmatrix} a_{00} & a_{01} & \ldots & a_{0n} & \ldots \\ a_{10} & a_{11} & \ldots & a_{1n} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k0} & a_{k1} & \ldots & a_{kn} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

whose elements a_{kn} satisfy following conditions
(i) $\quad a_{kn} \to 0 \quad (k \to \infty, \geq n0),$

(ii) $\quad |a_{k0}| + |a_{k1}| + \ldots + |a_{kn}| < k ,\quad (n, \ \geq 0, k \geq 0 \ \ k$ *fixed)*

(iii) $\sum_{n=0}^{\infty} a_{kn} = A_k \to 1$ *(k→∞),*

then φ(x) is one solution of (1) if

$|F(x)| \leq M$ *,* $x \in [a, b]$ *;* $\lim_{x\to b} F(x) = F(b)$ *, where F(x) is bounded in [a, b], continous from left in b.*

c) If series (3) is C_k*-sumabile with the sum φ(x), then φ(x) is one solution of (1).*

d) If series (3) is A-sumabile with the sum φ(x), then φ(x) is one solution of (1).

Note that considering cases c) and d) is justified since there is no assumptions about sumability of the F(x).

Proof. a) The statement can be proved directly by changing φ(x) in (1) by series (3), that is

$$\varphi(x) = \frac{1}{2}F(b) + \sum_{\nu=0}^{\infty}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right) - F(b)\right]$$

$$\varphi\left(f(x)\right) = \frac{1}{2}F(b) + \sum_{\nu=0}^{\infty}(-1)^{\nu}\left[F\left(f^{\nu+1}(x)\right) - F(b)\right]$$

$$\varphi(x) + \varphi\left(f(x)\right) = F(b) + \lim_{n\to\infty}\sum_{\nu=0}^{n}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right) - F(b)\right] + \lim_{n\to\infty}\sum_{\nu=0}^{n}(-1)^{\nu}\left[F\left(f^{\nu+1}(x)\right) - F(b)\right]$$

$$= F(b) + F(x) - F(b) + \lim_{n\to\infty}\sum_{\nu=1}^{n}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right) - F(b)\right] + \lim_{n\to\infty}\sum_{\nu=1}^{n+1}(-1)^{\nu-1}\left[F\left(f^{\nu}(x)\right) - F(b)\right] =$$

$$= F(x) + \lim_{n\to\infty}\sum_{\nu=1}^{n}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right) - F(b)\right] - \lim_{n\to\infty}\sum_{\nu=1}^{n}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right) - F(b)\right] = F(x).$$

b) By putting

$$s_n(x) = \frac{1}{2}F(b) + \sum_{\nu=0}^{n}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right) - F(b)\right] \quad (n = 0, 1, 2, \ldots)$$

$$s_k'(x) = \sum_{n=0}^{\infty} a_{kn} s_n(x) \quad (k = 0, 1, 2, \ldots)$$

and taking in the account assumption on T-sumability of the series (3), it gives

$$\varphi(x) = \lim_{k\to\infty} s_k'(x)$$

On the other hand, it can be easily shown that

$$s_n\left(f(x)\right) = F(x) - s_n(x) - (-1)^{n+1}\left[F\left(f^{n+1}(x)\right) - F(b)\right]$$

Namely,

$$s_n\left(f(x)\right) = \frac{1}{2}F(b) + \sum_{\nu=0}^{n}(-1)^{\nu}\left[F\left(f^{\nu+1}(x)\right) - F(b)\right],$$

so

$$s_n\big(f(x)\big)-s_n(x)=F(b)+\sum_{\nu=0}^{n}(-1)^{\nu}\Big[F\big(f^{\nu+1}(x)\big)-F(b)\Big]+\sum_{\nu=0}^{n}(-1)^{\nu}\Big[F\big(f^{\nu}(x)\big)-F(b)\Big]=$$

$$=F(b)+\sum_{\nu=1}^{n+1}(-1)^{\nu-1}\Big[F\big(f^{\nu}(x)\big)-F(b)\Big]+F(x)-F(b)+\sum_{\nu=1}^{n}(-1)^{\nu}\Big[F\big(f^{\nu}(x)\big)-F(b)\Big]=$$

$$F(x)-\sum_{\nu=1}^{n}(-1)^{\nu}\Big[F\big(f^{\nu}(x)\big)-F(b)\Big]-(-1)^{n+1}\Big[F\big(f^{n+1}(x)\big)-F(b)\Big]+\sum_{\nu=1}^{n}(-1)^{\nu}\Big[F\big(f^{\nu}(x)\big)-F(b)\Big]$$

$$=F(x)-(-1)^{n+1}\Big[F\big(f^{n+1}(x)\big)-F(b)\Big],$$

i.e.

$$s_n\big(f(x)\big)=F(x)-s_n(x)-(-1)^{n+1}\Big[F\big(f^{n+1}(x)\big)-F(b)\Big]$$

Now, it is

$$s_k'\big(f(x)\big)=\sum_{n=0}^{\infty}a_{kn}s_n\big(f(x)\big)=$$

$$=\sum_{n=0}^{\infty}a_{kn}F(x)-\sum_{n=0}^{\infty}a_{kn}s_n(x)-\sum_{n=0}^{\infty}(-1)^{n+1}a_{kn}\Big[F\big(f^{n+1}(x)\big)-F(b)\Big]=$$

$$=A_kF(x)-s_k'(x)-\sum_{n=0}^{\infty}(-1)^{n+1}a_{kn}\Big[F\big(f^{n+1}(x)\big)-F(b)\Big].$$

i.e.

$$s_k'\big(f(x)\big)=A_kF(x)-s_k'(x)-\sum_{n=0}^{\infty}(-1)^{n+1}a_{kn}\Big[F\big(f^{n+1}(x)\big)-F(b)\Big]. \qquad (5)$$

Last term on the right side of relation (5) can be evaluated on following way

$$\Big|\sum_{n=0}^{\infty}(-1)^{n+1}a_{kn}\Big[F\big(f^{n+1}(x)\big)-F(b)\Big]\Big|$$

$$\le\sum_{n=0}^{N}|a_{kn}|\Big|F\big(f^{n+1}(x)\big)-F(b)\Big|+\sum_{n=N+1}^{\infty}|a_{kn}|\Big|F\big(f^{n+1}(x)\big)-F(b)\Big|$$

$$\le 2M\frac{\varepsilon}{4M}+\frac{\varepsilon}{2}=\varepsilon \quad (k\ge k_0),$$

where fixed number N is chosen big enough so that

$$\big|F(f^{n+1}(x))-F(b)\big|\le\frac{\varepsilon}{2k} \quad (n\ge N+1),$$

and with such number N fixed, whole number $k_0 \geq 0$ chosen so that

$$\left|a_{kn}\right| \leq \frac{\varepsilon}{4M(N+1)} \qquad (k \geq k_0,\ n = 0, 1, \ldots, N).$$

Now, for $k \to \infty$ relation (5) takes form

$$\lim_{k\to\infty} s_k{}'\left(f(x)\right) = \lim_{k\to\infty} A_k F(x) - \lim_{k\to\infty} s_k{}'(x) - \lim_{k\to\infty} \sum_{n=0}^{\infty} (-1)^{n+1} a_{kn} \left[F\left(f^{n+1}(x)\right) - F(b)\right]$$

i.e.

$$\varphi(f(x)) = F(x) - \varphi(x).$$

c) Using well known fact

$$C_n^{(k)} = \frac{S_n^{(k)}}{\binom{n+k}{k}} \qquad (6)$$

let us prove that

$$C_n^{(k)}(x) = C_{n-1}^{(k)}(x) \cdot \frac{n}{n+k} + \frac{S_n^{(k-1)}(x)}{\binom{n+k}{k}} \qquad (7)$$

$$C_n^{(k)}(x) + C_n^{(k)}\left[f(x)\right] = F(x) - \frac{1}{2} F(b) \frac{k}{n+k} + \frac{S_n^{(k-1)}\left(f(x)\right)}{\binom{n+k}{k}}\ . \qquad (8)$$

From (6) follows

$$C_n^{(k)}(x) = \frac{S_n^{(k)}(x)}{\binom{n+k}{k}} = \frac{S_0^{(k-1)}(x) + S_1^{(k-1)}(x) + \ldots + S_{n-1}^{(k-1)}(x) + S_n^{(k-1)}(x)}{\binom{n+k}{k}} =$$

$$= \frac{S_0^{(k-1)}(x) + \ldots + S_{n-1}^{(k-1)}(x)}{\binom{n+k-1}{k} \cdot \frac{n+k}{n}} + \frac{S_n^{(k-1)}(x)}{\binom{n+k}{k}} =$$

$$= C_{n-1}^{(k)}(x) \cdot \frac{n}{n+k} + \frac{S_n^{(k-1)}(x)}{\binom{n+k}{k}}\ .$$

The equality (7) is proved.

Let us prove (8).
Adding

$$C_n^{(k)}(x) = C_{n-1}^{(k)}(x)\frac{n}{n+k} + \frac{S_n^{(k-1)}(x)}{\binom{n+k}{k}}$$

and

$$C_n^{(k)}\big(f(x)\big) = C_{n-1}^{(k)}\big(f(x)\big)\cdot\frac{n}{n+k} + \frac{S_n^{(k-1)}\big(f(x)\big)}{\binom{n+k}{k}}$$

gives

$$C_n^{(k)}(x) + C_n^{(k)}\big(f(x)\big) = \left\{C_{n-1}^{(k)}(x) + C_{n-1}^{(k)}\big(f(x)\big)\right\}\frac{n}{n+k} + \frac{S_n^{(k-1)}(x)}{\binom{n+k}{k}} + \frac{S_n^{(k-1)}\big(f(x)\big)}{\binom{n+k}{k}} =$$

$$= \frac{n}{n+k}\cdot\frac{S_{n-1}^{(k)}(x)}{\binom{n+k-1}{k}} + \frac{S_n^{(k-1)}(x)}{\binom{n+k}{k}} + \frac{S_n^{(k-1)}\big(f(x)\big)}{\binom{n+k-1}{k}}\cdot\frac{n}{n+k} + \frac{S_n^{(k-1)}\big(f(x)\big)}{\binom{n+k}{k}} \; .$$

The sum of the first three addents is equal to

$$A = \frac{S_{n-1}^{(k)}(x)}{\binom{n+k-1}{k}\frac{n+k}{k}} + \frac{S_n^{(k-1)}(x)}{\binom{n+k}{k}} + \frac{S_{n-1}^{(k)}\big(f(x)\big)}{\binom{n+k-1}{k}\frac{n+k}{k}} =$$

$$\frac{1}{\binom{n+k}{k}}\left\{S_{n-1}^{(k)}(x) + S_n^{(k-1)}(x) + S_{n-1}^{(k)}\big(f(x)\big)\right\} =$$

$$= \frac{1}{\binom{n+k}{k}}\left\{S_{n-1}^{(k)}(x) + S_n^{(k)}(x) - S_{n-1}^{(k)}(x) + S_{n-1}^{k}\big(f(x)\big)\right\} = \frac{1}{\binom{n+k}{k}}\left\{S_n^{(k)}(x) + S_{n-1}^{(k)}\big(f(x)\big)\right\} =$$

$$\frac{1}{\binom{n+k}{k}}\left\{\binom{n+k-1}{k-1}s_0(x) + \ldots + \binom{k-1}{k-1}s_n(x) + \binom{n+k-2}{k-1}s_0\big(f(x)\big) + \ldots + \binom{k-1}{k-1}s_{n-1}\big(f(x)\big)\right\}$$

$$= \frac{1}{\binom{n+k}{k}}\left\{\binom{n+k-1}{k-1}\big[s_1(x) + s_0\big(f(x)\big)\big] + \ldots + \binom{k-1}{k-1}\big[s_n(x) + s_{n-1}\big(f(x)\big)\big] + \binom{n+k-1}{k-1}s_0(x)\right\}$$

Using previously given relations

$$s_n(x)=\frac{1}{2}F(b)+\sum_{\nu=0}^{n}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right)-F(b)\right]$$

$$s_{n-1}\left(f(x)\right)=F(x)-s_{n-1}(x)-(-1)^{n}\left[F\left(f^{n}(x)\right)-F(b)\right]$$

it follows

$$s_n(x)+s_{n-1}(x)=F(x)-\left\{\frac{1}{2}F(b)+\sum_{\nu=0}^{n-1}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right)-F(b)\right]\right\}-(-1)^{n}\left[F\left(f^{n}(x)\right)-F(b)\right]+$$

$$+\frac{1}{2}F(b)+\sum_{\nu=0}^{n-1}(-1)^{\nu}\left[F\left(f^{\nu}(x)\right)-F(b)\right]+(-1)^{n}\left[F\left(f^{n}(x)\right)-F(b)\right]=F(x)\quad .$$

Now, by applying well known features of binomial coeficients in the sum A we get

$$A=\frac{1}{\binom{n+k}{k}}\left\{\binom{n+k-2}{k-1}F(x)+\ldots+\binom{k-1}{k-1}F(x)+\binom{n+k-1}{k-1}\left(\frac{1}{2}F(b)+F(x)-F(b)\right)\right\}$$

$$=-\frac{\binom{n+k-1}{k-1}}{\binom{n+k}{k}}\cdot\frac{1}{2}F(b)+F(x)\cdot\frac{\binom{n+k-1}{k-1}+\binom{n+k-2}{k-1}+\ldots+\binom{k-1}{k-1}}{\binom{n+k}{k}}=$$

$$=\frac{1}{2}F(b)\cdot\frac{k}{n+k}+F(x)\quad . \qquad (9)$$

Hence, from (9) it follows

$$C_n^{(k)}(x)+C_n^{(k)}\left(f(x)\right)=F(x)-\frac{1}{2}F(b)\cdot\frac{k}{n+k}+\frac{S_n^{(k-1)}\left(f(x)\right)}{\binom{n+k}{k}}$$, i.e. the equality (8).

Leting n → ∞, and taking in account assumption on C_k-sumability of the series (3), it follows

$$\lim_{n\to\infty}C_n^{(k)}(x)=\varphi(x)\quad ,$$

$$\lim_{n\to\infty}C_n^{(k)}\left(f(x)\right)=\varphi\left(f(x)\right)\quad ,$$

$$\lim_{n\to\infty}\frac{1}{2}F(b)\cdot\frac{k}{n+k}=0\quad ,$$

$$\lim_{n\to\infty}\frac{S_n^{(k-1)}\left(f(x)\right)}{\binom{n+k}{k}}=0\quad ,$$

i.e.

$$\varphi(x)+\varphi\big(f(x)\big)=F(x) \ .$$

d) Puting

$$\sigma(x,t)=\frac{1}{2}F(b)+\sum_{\nu=0}^{\infty}(-1)^{\nu}\Big[F\big(f^{\nu}(x)\big)F(b)\Big]t^{\nu}$$

$$\varphi(x)=\lim_{t\to 1}\sigma(x,t)$$

and taking in the account assumptions of the Theorem 2 as well as the assumptions given before Theorem 2, it can be easily calculated that

$$\sigma\big(f(x),t\big)+\sigma(x,t)=F(b)+\sum_{\nu=0}^{\infty}(-1)^{\nu}\Big[F\big(f^{\nu+1}(x)\big)-F(b)\Big]t^{\nu}+\sum_{\nu=0}^{\infty}(-1)^{\nu}\Big[F\big(f^{\nu}(x)\big)-F(b)\Big]t^{\nu}$$

$$=F(x)+\left(1-\frac{1}{t}\right)\sum_{\nu 01}^{\infty}(-1)^{\nu}\Big[F\big(f^{\nu}(x)\big)-F(b)\Big]t^{\nu}$$

Let t↑1. Then, by assumption on A-sumability of the series (3), series

$$\sum_{\nu=1}^{\infty}(-1)^{\nu}\Big[F\big(f^{\nu}(x)-F(b)\big)\Big]t^{\nu}$$

converges for t↑1, thus

$$\varphi(x)+\varphi\big(f(x)\big)=F(x) \ .$$

Theorem is proved.

Next theorem, which is proved in Pjanic(2007), will be needed in the following part of paper.

THEOREM 3. *Given equation (1) where F(x) is continuous and f(x) continuous and strictly increasing in [a,b]. Then (1) has at most one solution continuous in [a,b) and at most one solution continuous in (a,b].*

It is easy now to prove next theorem.

THEOREM 4. *Given equation (1) where F(x) be continuous and f(x) continuous and strictly increasing in [a,b]. If series (3) is T-sumabile with the sum φ(x) continuous in (a, b], then series (3) is convergent in (a,b] with the same sum φ(x). Function φ(x) is the only continuous solution of the equation (1) in (a,b].*

Proof. According to Theorem 2b) function φ(x) is one solution of the equation (1) in (a,b]. The solution φ(x) is continuous in (a,b], by assumption.

Now, from Theorem 3 and Theorem 1 it follows that φ(x), given by (3), is the only continuous solution of equation (1) in (a,b].

THEOREM 5. *Given equation (1) where F(x) is continuous and f(x) is continuous and strictly increasing in [a,b]. If*

$$F(x)\ge F\big[f(x)\big], \qquad x\in\big[b-\eta,b\big]\eta>0 \tag{10}$$

or

$$F(x)\le F\big[f(x)\big], \qquad x\in\big[b-\eta,b\big]\eta>0 \ , \tag{11}$$

then the only one continuous solution of equation (1) in (a,b] neccessarily exists.

Proof. Suppose that (8) is satisfied.

It is sufficient to prove that series (3) is uniformely convergent in interval [h,b] , where $h \in (a,b)$ is arbitrary fixed element.

Put $h_\nu = f^\nu(h)$. Then, there is number N so that

$$b-\eta \le h_\nu < b \qquad (\nu \ge N)$$

According to (10) it is

$$A_\nu = \sup_{[h_\nu, h_{\nu+1}]} F(x) \ge \sup_{[h_{\nu+1}, h_{\nu+2}]} F(x) = A_{\nu+1} \qquad (\nu \ge N) \ .$$

If $x \in [h,b]$, then it can be always founded whole number k so that

$$x_\nu = f^\nu(x) \in [h_{k+\nu}, h_k]$$

Thus,

$$F(x_\nu) - F(b) \ge F(x_{\nu+1}) - F(b) \ge 0 \qquad (\nu \ge N),$$

and

$$\left|\sum_{\nu=n}^{\infty} (-1)^\nu [F(x_\nu) - F(b)]\right| \le F(x_n) - F(b) \le A_{n+k} - F(b) \le A_n - F(b) < \varepsilon \quad (n \ge N_1 \ge N)$$

as $A_\nu \to F(b)$ ($\nu \to \infty$).

The last evaluation gives uniform convergence of the series (3) in [h,b].

If the condition (11) is satisfied then the statement of the Theorem 5 can be obtained by changing F(x) with –F(x) and reapiting procedure.

In the proof Theorem 6 will be used next theorem proved by Knopp. Knopp's theorem will be given without proof.

Theorem (Knopp): *Series* $\sum a_n$ *with partial sums* s_n *is* C_1*-sumabile with sum s, if and only if series*

$$\text{(A)} \quad \sum_{\nu=0}^{\infty} \frac{a_\nu}{\nu+1}$$

is convergent and its reminder

$$\rho_n = \frac{a_{n+1}}{n+2} + \frac{a_{n+2}}{n+3} + \ldots \quad (n=0,\ 1,2,\ldots)$$

satisfies

$$\text{(B)} \quad s_n + (n+1)\rho_n \to s.$$

If σ_n *denote partial sums of (A), and if* σ *denotes sum of (A), then*

$$\text{(B!)} \quad s - s_n - (n+1)(\sigma - \sigma_n) \to 0.$$

Now, Theorem 6 can be given and proved.

THEOREM 6. *Given equation (1) where F(x) is continuous and f(x) is continuous and strictly increasing in [a,b]. Let G(x) = F(x) – F(b). If*

(i) $a_n = f^n(a_0)$, *where* a_0 *is fixed,* $0 < a_0 < b$,

(ii) $0 < \omega G(x) \le \dfrac{A}{n^\alpha}$, $x \in [a_n, a_{n+1})$, $n > n_0 > 0$,

(iii) $\dfrac{G[f(x)]}{G(x)} \le 1 + \dfrac{1}{n}$, $\omega = 1$ *or* $\omega = -1$

where A and α are positive constants, then equation (1) has the only one continuous solution in (a,b].

Proof. Suppose that $\omega = 1$.

Note that Theorem 6 stays valid if for the starting interval $[a_0,a_1)$ one takes interval $[a_k,a_{k+1})$, (for k whole number), and changing at the same time n_0 by $n_0 - k$.

Now, it has to be proved following:

1° Alternated series

$$\sum_{\nu=1}^{\infty} (-1)^\nu \frac{G[f^\nu(x)]}{\nu} \tag{12}$$

converges apsolutely and uniformely in $[a_k,b]$, (k whole number).

2° Sequence

$$c_n(x) = \frac{1}{2}F(b) + \sum_{\nu=0}^{n} (-1)^\nu G[f^\nu(x)] + (n+1)\sum_{\nu=n+1}^{\infty} (-1)^\nu \frac{G[f^\nu(x)]}{\nu} \tag{13}$$

converges uniformely in $[a_0,b]$ when $n \to \infty$.

Taking in the account previous note, it is sufficient to prove 1° in the case k = 0.

If $x \in [a_n, a_{n+1})$ $(n \geq 0)$, put $x = f^n(x_0) = x_n$, $x_0 \in [a_0, a_1)$ that, using (ii), gives,

$$0 < \frac{G\left[f^{\nu}(x)\right]}{\nu} = \frac{G\left[f^{n+\nu}(x_0)\right]}{\nu} = \frac{G(x_{n+\nu})}{\nu} \leq \frac{A}{\nu\,(n+\nu)^{\alpha}} \leq \frac{A}{\nu^{1+\alpha}}$$

By this 1° is proved.

In order to prove 2° put

$$x = f^k(x_0) = x_k \in [a_k, a_{k+1}) , \ x_0 \in [a_0, a_1).$$

Then (13) implies

$$\left|c_{n+p}(x_k) - c_n(x_k)\right| = \left| \sum_{\nu=n+1}^{n+p} (-1)^{\nu} G(x_{\nu+k}) \left(1 - \frac{n+1}{\nu}\right) + p \sum_{\nu=n+p+1}^{\infty} (-1)^{\nu} \frac{G(x_{\nu+k})}{\nu} \right| =$$

$$= \left| \sum_{\mu=0}^{p-1} (-1)^{\mu+n+1} G(x_{\mu+n+1+k}) \left(1 - \frac{n+1}{\mu+n+1}\right) + p \sum_{\nu=n+p+1}^{\infty} (-1)^{\nu} \frac{G(x_{\nu+k})}{\nu} \right| \leq$$

$$\leq \left| \sum_{\mu=0}^{p-1} (-1)^{\mu+n+1} G(x_{\mu+n+1+k}) \left(1 - \frac{n+1}{\mu+n+1}\right) \right| + \left| p \sum_{\nu=n+p+1}^{\infty} (-1)^{\nu} \frac{G(x_{\nu+k})}{\nu} \right| \quad (14)$$

Simple transformations of the term under first absolute value sing in the right side of (14) give

$$\sum_{\mu=0}^{p-1} (-1)^{\mu+n+1} G(x_{\mu+n+1+k}) \left(1 - \frac{n+1}{\mu+n+1}\right) = \sum_{i=1}^{p-1} \left[\sum_{\mu=i}^{p-1} (-1)^{\mu+n+1} \frac{G(x_{\mu+n+1+k})}{\mu+n+1} \right]$$

Furthermore, considering

$$\frac{G(f(x))}{G(x)} \leq 1 + \frac{1}{n} \ , \text{ i.e. } \quad \frac{G(f(x))}{n+1} \leq \frac{G(x)}{n} \ ,$$

the term under the second apsolute value sign on the right side of (14) is less then

$$p \cdot \frac{G(x_{n+p+1+k})}{n+p+1} .$$

Now, relation (14) takes form

$$\left|c_{n+p}(x_k) - c_n(x_k)\right| \leq \sum_{i=1}^{p-1} \left| \sum_{\mu=i}^{p-1} (-1)^{\mu+n+1} \frac{G(x_{\mu+n+1+k})}{\mu+n+1} \right| + p \cdot \frac{G(x_{n+p+1+k})}{n+p+1}$$

$$\sum_{i=1}^{p-1} \left| \sum_{\mu=i}^{p-1} (-1)^{\mu+n+1} \frac{G(x_{\mu+n+1+k})}{\mu+n+1} \right| + p \cdot \frac{G(x_{n+p+1+k})}{n+p+1}$$

Taking into account that

$$\frac{G(x_{\mu+n+1+k})}{\mu+n+1} \leq \frac{1}{\mu+n+1} \cdot \frac{A}{(\mu+n+1)^{\alpha}} = \frac{A}{(\mu+n+1)^{1+\alpha}}$$

from it finally gives

$$\left| c_{n+p}(x_k) - c_n(p) \right| < \varepsilon$$

for every $n \geq n_0$ that is large enough, for every p = 0, 1, 2,... and for every $x \in [a_0, b]$.

Thus, 2° is proved.

Puting

$$\varphi(x) = \lim_{n\to\infty} c_n(x) \ , \ x \in [a_0, b]$$

and applying Knopp's Theorem that gives neccessary and sufficient conditions for C_1-sumability of (9), and using Theorem 2c) and Theorem 3, it follows that function φ(x) is the only continuous solution of (1) in [a,b].

For $\omega = -1$ the proof can be conducted in similar way, putting -F(x) instead of F(x).

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