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Some examples of contractions on C[0,1], $L^2(0,1)$ and R^n

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Abstract

This study proves that some operators constructed on the spaces C[0,1], $L^2(0,1)$ and R^n are contractions by using Banach fixed point theorem and the map contraction principle.

Key words: contraction, complete metric space, norm, Banach fixed point theorem.

Ejemplo de las contracciones en C[0,1], $L^2(0,1)$ y R^n

Resumen

Mediante el uso del teorema del punto fijo de Banach y el principio de la contracción del mapa, se prueba que algunos operadores construidos en los espacios C[0,1], $L^2(0,1)$ y R^n son contracciones.

Palabra clave: contracción, espacio métrico completo, norma, teorema de Banach de punto fijo.

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Introduction

Contractions on metric spaces and Banach fixed point theorem and its generalisations are widely discussed. Textbooks of Aljancic (1968), Kurepa (1990), Vajzovic (1971), and recent papers of Imdad and Khan (2006), Bollenbacher and Hicks (1988), are few examples of treating this topic.

However, examples of contractions on complete metric spaces and on normed spaces are usually omitted in textbooks. In this paper three contractions will be constructed on different normed spaces.

DEFINITION 1. Let X be metric space and A operator acting from X into X. Operator A is called contraction (in X) if there exist constant q, $0 \leq q < 1$, such that

$$\rho(Ax, Ay) \leq q\rho(x, y)$$

for all x,y $\in$ X.

Taking into account that in every normed space it is possible to introduce metric by the following relation

$$\rho(x, y) = \|x - y\|, \quad x,y \in X,$$

we can give examples of contractions on the normed spaces C[0,1] and $L^2(0,1)$.

Example 1. *Mapping A : C[0,1] → C[0,1] given with*

$$Ax(t) = \lambda x(t^{\beta}), \ \beta \geq 0$$

is contraction if and only if $|\lambda| < 1$.

Namely, evaluation of the norm

$$|Ax(t)| \leq |\lambda| \cdot |x(t^{\beta})| \leq |\lambda| \cdot \max_{0 \leq t \leq 1} |x(t)| = |\lambda| \cdot \|x\|$$

gives

$$\|Ax\| \leq |\lambda| \cdot \|x\|,$$

i.e.

$$\|A\| \leq |\lambda|. \tag{1}$$

On the other hand, for x(t) ≡ 1 it is $\|x\| = 1$ and Ax(t) ≡ λ. Hence, $\|Ax\| = |\lambda|$ and therefore

$$\|A\| = |\lambda|. \tag{2}$$

From (1) and (2) follows that $|\lambda| < 1$ if and only if A is contraction.

Example 2. *Maping A : $L^2(0,1) \to L^2(0,1)$ given with*

$$Ax(t) = \lambda x(t^{\beta}), \quad 0 < \beta \leq 1$$

is contraction if and only if $|\lambda| < \sqrt{\beta}$.

Namely, from $Ax(t) = \lambda x(t^{\beta})$ it is

$$|Ax(t)|^2 = |\lambda|^2 \cdot |x(t^{\beta})|^2 .$$

Integration of both sides of previous equality, in borders 0 to 1, gives

$$\int_0^1 |Ax(t)|^2 dt = |\lambda|^2 \int_0^1 |x(t^{\beta})|^2 dt$$

By puting $t^{\beta} = s$, last equality can be written as

$$\int_0^1 |Ax(t)|^2 dt = |\lambda|^2 \int_0^1 |x(s)|^2 \frac{1}{\beta} s^{\frac{1}{\beta}-1} ds \leq \frac{|\lambda|^2}{\beta} \int_0^1 |x(s)|^2 ds,$$

as $\frac{1}{\beta} - 1 \geq 0$.

Finally, by taking second root of both sides of previous equality, it can be obtained

$$\|Ax\|_{L^2} \leq \frac{|\lambda|}{\sqrt{\beta}} \|x\|_{L^2} ,$$

i.e.

$$\|A\| \leq \frac{|\lambda|}{\sqrt{\beta}}.$$

On the other hand, one can observe function $x(t) = t^a$, $a > 0$.

Now, it is

$$\|x\|_{L^2} = \sqrt{\int_0^1 t^{2a} dt} = \frac{1}{\sqrt{2a+1}} \quad (3)$$

and

$$Ax(t) = \lambda t^{a\beta} \quad ,$$

$$\|Ax\|_{L^2} = \sqrt{\int_0^1 |\lambda|^2 t^{2a\beta} dt} = \frac{|\lambda|}{\sqrt{2a\beta+1}} \quad . \quad (4)$$

Expressions (3) and (4) imply that

$$\|A\| \geq \frac{\|Ax\|}{\|x\|} = |\lambda| \sqrt{\frac{2a+1}{2a\beta+1}} \, .$$

Taking limit when $a \to +\infty$ results with

$$\|A\| \geq \frac{|\lambda|}{\sqrt{\beta}} \, .$$

Therefore, our statement holds.

Theorem 1. (Banach fixed point theorem). *If X is complete metric space and A contraction on X, then there exist one and only one solution* $\bar{x}$ *of the equation*

$$x = Ax, \quad (5)$$

i.e. contraction A has one and only one fix point.

Proof. Let $x_0 \in X$ and put

$x_1 = Ax_0$

$x_2 = Ax_1 = A(Ax_0) = A^2x_0$

.....

$x_n = Ax_{n-1} = \ldots = A^n x_0$

.....

Let us prove that $\{x_n\}$ is Cauchy sequence.

First at all, it is

$$\rho(x_n,x_{n-1}) = \rho(Ax_{n-1},Ax_{n-2}) \leq q\rho(x_{n-1},x_{n-2}).$$

By using previous inequality for n times, it can be concluded that

$$\rho(x_n,x_{n-1}) \leq q^{n-1}\rho(x_1,x_0). \quad (6)$$

For m > n, by using (6) follows

$$\rho(x_m,x_n) \leq \rho(x_m,x_{m-1}) + \rho(x_{m-1},x_{m-2}) + \ldots + \rho(x_{n+1},x_n) \leq$$

$$\leq (q^{m-1} + q^{m-2} + \ldots + q^n)\rho(x_1,x_0) =$$

$$= q^n \frac{1-q^{m-n}}{1-q} \rho(x_1,x_0) \leq \frac{q^n}{1-q} \rho(x_1,x_0) \, .$$

Hence,

$$\rho(x_m,x_n) \leq \frac{q^n}{1-q} \rho(x_1,x_0) \ (m>0) \quad (7)$$

From (7) follows that $\{x_n\}$ is Cauchy sequence.

Existence. As X is complete space, there exist $\bar{x} \in X$ such that

$$x_n \to \bar{x} \ (n \to \infty).$$

Now, it can be written

$$0 \leq \rho(\bar{x}, A\bar{x}) = \lim_{n\to\infty} \rho(x_{n+1}, A\bar{x}) = \lim_{n\to\infty} \rho(Ax_n, A\bar{x}) \leq q \lim_{n\to\infty} \rho(x_n, \bar{x}) = 0,$$

i.e.

$$\rho(\bar{x}, A\bar{x}) = 0$$

and

$$\bar{x} = A\bar{x}$$

The point $\bar{x} \in X$ is the fix point of the contraction A.

Uniquenes. If $\bar{\bar{x}} \in X$ is also fix point of the contraction A, then it is

$$\bar{\bar{x}} = A\bar{\bar{x}} \, .$$

So,

$$0 \leq \rho(\bar{x}, \bar{\bar{x}}) = \rho(A\bar{x}, A\bar{\bar{x}}) \leq q\rho(\bar{x}, \bar{\bar{x}}) \, ,$$

as $0 \leq q < 1$.

Previous inequalities are possible only if

$$\rho(\bar{x}, \bar{\bar{x}}) = 0 \qquad \text{i.e.} \quad \bar{x} = \bar{\bar{x}} \, .$$

The theorem is proved.

The following theorems that are derived from Theorem 1 can be proved by using the same algorithm given in the proof of Theorem 1 (see Vajzovic, 1971)).

Theorem 2. *If X is complete metric space, F $\subset X$ nonempty closed set in X and A contraction from F into F, then contraction A has one and only one fix point. That point is element of F.*

Theorem 3. *Let X is complete metric space and A operator form X into X. If there exist $n \in N$ such that A^n is contraction, A has one and only one fix point in X.*

Next statement can be proved by using Banach theorem and the contraction mapping principle.

STATEMENT 1. *Let matrix A has real and mutualy different eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$ with eigenvectors $e_1, e_2, \ldots, e_n$. If*

$$f(x) = \frac{Ax}{\|Ax\|}$$

is mapping defined in R^n on the unit sphere $\|x\| = 1$, then mapping f is contraction in some surrounding of e_1.

Namely, mapping f can be directly calculated in coordiantes:

$$f(x_1,\ldots,x_n) = \left(\frac{\lambda_1 x_1}{\sqrt{\lambda_1^2 x_1^2 + \ldots + \lambda_n^2 x_n^2}}, \ldots, \frac{\lambda_n x_n}{\sqrt{\lambda_1^2 x_1^2 + \ldots + \lambda_n^2 x_n^2}} \right),$$

i.e. mapping f can be comprehensed as n real functions with n variables given with

$$f_j(x_1,\ldots,x_n) = \frac{\lambda_j x_j}{\sqrt{\sum_{i=1}^{n} \lambda_i^2 x_i^2}}.$$

It is easy to show that

$$\frac{\partial f_j}{\partial x_i} = \frac{-\lambda_j \lambda_i^2 x_i x_j}{\left(\sum_{i=1}^{n} \lambda_i^2 x_i^2 \right)^{\frac{3}{2}}} \quad \text{for } i \neq j$$

and

$$\frac{\partial f_j}{\partial x_i} = \frac{\lambda_j (\lambda_1^2 x_2^2 + \ldots + \lambda_j^2 \hat{x}_j^2 + \ldots + \lambda_n^2 x_n^2)}{\left(\sum_{i=1}^{n} \lambda_i^2 x_i^2 \right)^{\frac{3}{2}}},$$ where $\wedge$ is „expelling“ sign.

As vector e_1 has coordinates (1,0,0,...,0), previous equations give

$$\frac{\partial f_j}{\partial x_i}(e_1) = 0 \quad \text{for } i \neq j,$$

$$\frac{\partial f_j}{\partial x_j}(e_1) = \frac{\lambda_j}{\lambda_1} \quad \text{for } j \neq 1,$$

$$\frac{\partial f_1}{\partial x_1}(e_1) = 0.$$

Therefore, differential of the function f in the point e_1 is equal to

$$df(e_1) = \begin{bmatrix} 0 & & & 0 \\ & \frac{\lambda_2}{\lambda_1} & & \\ & & \ddots & \\ 0 & & & \frac{\lambda_n}{\lambda_1} \end{bmatrix}.$$

Taking the norm one gets

$$\|df(e_1)x\| \leq \frac{\lambda_2}{\lambda_1}\|x\|,$$

i.e.

$$\|df(e_1)\| \leq \frac{\lambda_2}{\lambda_1}.$$

Since $\frac{\partial f_j}{\partial x_i}$ are continuous functions, there is q such that $\frac{\lambda_2}{\lambda_1} < q < 1$, and there is convex surounding U of the point e_1 so that

$$\forall x \in U, \quad \|df(x)\| \leq q.$$

Applying the theorem of the mean value gives

$$\|f(x) - f(y)\| \leq \sup_{0<\theta<1} \|df(\theta x + (1-\theta)y)\| \cdot \|x - y\| \leq q\|x - y\|,$$

i.e. f is contraction.

REMARK. In the previous observation, domain of f is widened on some surounding of e_1 wich contains even points outside the unit sphere. This was correct as $f(x) = \frac{Ax}{\|Ax\|}$ is defined for all $x \neq 0$

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